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Two-dimensional integrable potentials with quartic invariants

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Abstract. For autonomous two-dimensional conservative dynamical systems we derive four necessary and sufficient conditions which the potential function $U(x, y)$ has to satisfy in order that it is integrable with the second constant of motion quartic in the velocity components. We also develop the method by means of which we find the quartic invariant for a given potential satisfying these conditions.

Certain degenerate cases leading to pseudo-quartic integrals are discussed. Two examples and a counter-example are presented.

1. Introduction

A basic result regarding integrability of potentials $V(x, y)$ for autonomous conservative dynamical systems of two degrees of freedom was obtained by Darboux [1] and was reproduced later by Whittaker [2]. It accounts for integrable systems possessing, besides the energy integral, a second constant of the motion, quadratic in the velocity components $\dot{x}, \dot{y}$. Darboux proved that these potentials must satisfy a second-order partial differential equation in $V(x, y)$ which he also solved in terms of two arbitrary functions. Certain special cases, omitted by Darboux, were studied much later [3].

Many papers have appeared recently which face the problem of constructing integrable dynamical systems (more often of two degrees of freedom) with a second constant which is usually algebraic in the velocity components or, at cases, of other prespecified form. A full account of relevant results may be found in a review by Hietarinta [4]. However, relatively few concrete results, similar to that of Darboux, for algebraic integrals of higher order, are found in the literature. In particular:

(i) Holt [5] found *two* nonlinear partial differential equations which the potential $V(x, y)$ has to satisfy so that a second integral, cubic in the momenta, does exist. These conditions include, apart from ten constants, an arbitrary function $\Psi(V)$. No general solution for these equations is known [5].

(ii) Considering third and fourth-order invariants and using complex conjugate variables, Kaushal *et al* [6] derived what they called 'potential equations'. For the case of quartic invariants (which is the subject of the present paper too) the potential equation given by Kaushal *et al* [6] involves potential derivatives up to the fourth order and also nine complex constants. This condition is necessary but not sufficient, although this fact is not clearly stated. On the contrary, the authors let the reader understand that, if the potential satisfies their condition (equation (3.6), p 423), the corresponding integral of motion can be found. As a matter of fact, they have taken care of the compatibility conditions for all pertinent coefficients appearing in the fourth and second-order powers of the momenta but not of the (zero order) last coefficient $a_0(x, y)$ (in our notation: $I(x, y)$).

(iii) Fokas and Lagerstrom [7] addressed the problem of integrals of motion at most cubic in the momenta. For cubic invariants they found a relation which is linear in the derivatives of the potential up to the third-order with coefficients which are also linear combinations of the coefficients d_{ijk} (and their first and second-order derivatives) of the corresponding third-order powers in the momenta in the expression of the second integral. This relation, however, is a necessary, not sufficient, condition for a potential $V(x, y)$ to admit such an integral of motion.

(iv) Fordy *et al* [8] have found some quartic integrals corresponding to quartic potentials with four degrees of freedom. Fordy [9] has also found a second quartic for specific values of the Hénon–Heiles model.

(v) Certain quartic integrals associated with third and fourth-degree polynomial potentials were established by Grammaticos *et al* [10] who used a direct approach combined with Painlevé analysis. Of much interest to the present paper is a statement of theirs according to which a quartic integral exists whenever the potential satisfies two partial differential equations. One of these equations includes the coefficients f_0, f_1, f_2, f_3, f_4 of the fourth degree in x, y in the expression of the second integral (which are known polynomials). The other relation, however, includes the coefficients of the second degree (which are not known). The authors find it convenient to restrict themselves to non-trivial cases with f_i $(i = 0, 1, \ldots, 4)$ constants.

(vi) Most of the results known in the literature for higher-order invariants were obtained after some sort of restriction made either on the form of the invariant or on the form of the potential. Thus, Leach [11], commenting on a paper by Thompson [11], re-examined invariants of the third and fourth-order with a leading term $(yp_x - xp_y)^3$ or $(yp_x - xp_y)^4$ correspondingly and found the form of the general solution for the potential. These results were completed by Sen [11] who found it convenient to use polar coordinates. In another paper, Sen [12] changed the ansatz for the nth order invariant by considering higher-order terms of the form $(xp_y - yp_x)^{n-2}(p_x^2 + p_y^2)$. For $n = 3$, 4 and 5 he offered, in complex coordinates, conditions for the corresponding potential. Very recently Evans [13] made a search for autonomous Hamiltonians of two degrees of freedom admitting a second integral quartic in the momenta with leading term $(\dot{x}^2\dot{y}^2)/2$. He expressed the potential as a sum of four functions u_1, u_2, u_3, u_4 and reproduced a functional relation between these functions, also given by Hietarinta [4]. Some of Evans' results reduce to simpler systems found earlier by Bozis [13] and some new integrable systems are found by the method of Lax pairs.

In the present paper we derive four necessary and sufficient conditions for a potential function $U(x, y)$ so that it admits an integral of motion quartic in the velocity components. The conditions include the 15 constants associated with quartic integrals [4] and derivatives of $U(x, y)$ with respect to x, y up to the fifth order. In fact each condition is checked as follows: given a potential function one finds 60 functions $H_i^{(k)}$ $(i = 1, \ldots, 15)$ for $k = 1, 2, 3, 4$ corresponding to each condition in terms of the given $U(x, y)$ and derivatives of it only. In general, if there exist 15 constants $c_1, \ldots, c_{15}$, not all zero, such that each sum $\Sigma_{i=1}^{15} c_i H_i^{(k)}$ vanishes identically for $k = 1, 2, 3, 4$ then $U(x, y)$ is integrable and the corresponding quartic invariant can be found.

At cases, it may be that the above quartic is the square of a quadratic constant, other than the energy integral E, but again integrability is established. For central potentials, the pseudo-quartic may be simply the fourth power C^4 of the angular momentum constant C or the constant EC^2.

2. The vanishing Poisson bracket in new variables

Dealing with autonomous dynamical systems of two degrees of freedom and using a new set of variables we derived, in a previous paper [14], an equation equivalent to the condition of a vanishing Poisson bracket. The variables used were suggested in the light of certain 'inverse problem considerations' and they were more effective in handling terms of the same degree in the velocity components appearing in second integrals of motion. Indeed, the use of these variables is crucial because it makes the lengthy calculations involved in this problem much shorter by reducing the number of unknown functions involved [14]. We comment on this point again in section 3.

Suppose that

$$\varphi(x, y, \dot{x}, \dot{y}) = c \tag{1}$$

is a second integral of motion, besides the energy integral

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - U(x, y) \tag{2}$$

corresponding to the potential function $U(x, y)$.

Let us adopt the notation

$$U_{ij} = \frac{\partial^{i+j}}{\partial x^i \partial x^j} U. \tag{3}$$

Thus, for instance, $U_{10} = U_x$, $U_{01} = U_y$, $U_{11} = U_{xy}$, $U_{12} = U_{xyy}$, etc., where subscripts x, y denote partial differentiation.

We introduce now the following transformation

$$x = x \qquad y = y \qquad \dot{x} = -\varepsilon z \left(\frac{U_{10} + zU_{01}}{\omega} \right)^{1/2} \qquad \dot{y} = \varepsilon \left(\frac{U_{10} + zU_{01}}{\omega} \right)^{1/2} \tag{4}$$

with $\varepsilon = \pm 1$. Then the position coordinates are unaltered, while the velocity components $\dot{x}$, $\dot{y}$ are expressed in terms of the new variables z and ω, where

$$z = -\frac{\dot{x}}{\dot{y}} \qquad \omega = \frac{U_{10}\dot{y} - U_{01}\dot{x}}{\dot{y}^3}. \tag{5}$$

The second integral (1) becomes

$$\Phi(x, y, z, \omega) = \varphi(x, y, \dot{x}, \dot{y}) \tag{6}$$

and the condition that the Poisson bracket $[E, \varphi]$ vanishes along any orbit is now [14]

$$z\Phi_x - \Phi_y + \omega\Phi_z + \omega(\omega L^* + M^*)\Phi_\omega = 0 \tag{7}$$

with

$$L^* = \frac{3U_{01}}{U_{10} + zU_{01}} \qquad M^* = \frac{z(U_{20} - U_{02}) + (z^2 - 1)U_{11}}{U_{10} + zU_{01}}. \tag{8}$$

3. Fourth-degree integrals of motion

As an application of formula (7) we shall study in this paper integrals of motion which are polynomials of the fourth degree in the velocity components $\dot{x}$, $\dot{y}$, i.e. integrals of the form

$$\varphi = A\dot{x}^4 + B\dot{x}^3\dot{y} + C\dot{x}^2\dot{y}^2 + D\dot{x}\dot{y}^3 + E\dot{y}^4 + F\dot{x}^2 + G\dot{x}\dot{y} + H\dot{y}^2 + I \tag{9}$$

corresponding to the potential function $U(x, y)$. The symmetry under time reversal of the Hamiltonian requires that $\varphi(x, y, \dot{x}, \dot{y})$ contains only even powers in the velocity components $\dot{x}, \dot{y}$. The nine coefficients appearing in the expression (9) are, in general, functions of the position coordinates x, y.

Transformed to the new variables x, y, z, ω, equation (9) becomes

$$\Phi(x, y, z, \omega) = (U_{10} + zU_{01})^2 f(x, y, z)\omega^{-2} + (U_{10} + zU_{01})g(x, y, z)\omega^{-1} + I(x, y) \tag{10}$$

where

$$f(x, y, z) = Az^4 - Bz^3 + Cz^2 - Dz + E \tag{11a}$$

$$g(x, y, z) = Fz^2 - Gz + H \tag{11b}$$

$$I(x, y) = I(x, y). \tag{11c}$$

Function $I(x, y)$ remained unaltered, of course. It is also seen that, due to the transformation used, the problem of determining the three functions F, G, H (of the two variables x, y) is now replaced by the problem of determining one function $g(x, y, z)$ (of three variables) which, though, is a second-degree polynomial in the independent variable z. Something analogous has happened with the five functions A, B, C, D, E, the determination of which now requires the knowledge of the unique function $f(x, y, z)$ which is actually a fourth-degree polynomial in z. As a matter of fact it will become evident shortly that there is no essential difficulty with the five coefficients A, B, C, D, E which are known to be fourth-degree polynomials in x, y.

The reasoning now goes as follows: for a given potential function $U(x, y)$, function $\Phi(x, y, z, \omega)$, given by (10), must satisfy equation (7). Arranging in powers $\omega^{-2}, \omega^{-1}, \omega^0$ of the independent variable ω we obtain the following equations correspondingly

$$zf_x - f_y = 0 \tag{12}$$

$$zg_x - g_y - 4U_{01}f + (U_{10} + zU_{01})f_z = 0 \tag{13}$$

$$(U_{10} + zU_{01})g_z - 2U_{01}g + zI_x - I_y = 0. \tag{14}$$

In view of $(11a)$, equation (12) gives

$$\begin{aligned} &A_x = 0 \qquad A_y + B_x = 0 \qquad B_y + C_x = 0 \\ &C_y + D_x = 0 \qquad D_y + E_x = 0 \qquad E_y = 0. \end{aligned} \tag{15}$$

The solution of system (15) is easily found [10]:

$$A = a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 \tag{16a}$$

$$B = -x(4a_4y^3 + 3a_3y^2 + 2a_2y + a_1) + (b_3y^3 + b_2y^2 + b_1y + b_0) \tag{16b}$$

$$C = x^2(6a_4y^2 + 3a_3y + a_2) - x(3b_3y^2 + 2b_2y + b_1) + (c_2y^2 + c_1y + c_0) \tag{16c}$$

$$D = -x^3(4a_2y + a_3) + x^2(3b_3y + b_2) - x(2c_2y + c_1) + (d_1y + d_0) \tag{16d}$$

$$E = a_4x^4 - b_3x^3 + c_2x^2 - d_1x + e_0 \tag{16e}$$

where the 15 constants

$$a_4,\ a_3,\ a_2,\ a_1,\ a_0;\ b_3,\ b_2,\ b_1,\ b_0;\ c_2,\ c_1,\ c_0;\ d_1,\ d_0;\ e_0 \tag{16*}$$

play a significant role in the analysis which follows. With the aid of the above constants we shall offer four conditions in order that a given potential function $U(x, y)$ is integrable with a second constant of motion of the form (9). Needless to say that the potential to be tested for integrability may generally be brought to a simpler form by an appropriate translation and rotation of the axes, as well as by an appropriate factorization.

If we suppose momentarily that function $I(x, y)$ is known we can find the general solution of (14) for $g = g(x, y, z)$. It is

$$g(x, y, z) = z^2 h + \left(\frac{2U_{10}h + I_x}{U_{01}}\right) z + \frac{U_{10}}{2U_{01}^2} I_x - \frac{1}{2U_{01}} I_y + \frac{U_{10}^2}{U_{01}^2} h \tag{17}$$

where $h(x, y)$ is an arbitrary function of x, y. Identifying function $g(x, y, z)$ as given by equations (17) and (11b) we can find explicitly the three coefficients F, G, H, provided that $I(x, y)$ exists and is known.

The question now is: Does function $I(x, y)$ exist? Besides, if $I(x, y)$ does exist, how can it be found?

4. Conditions for the existence of the function $I(x, y)$

We shall see that $I(x, y)$ must satisfy a system of three linear partial differential equations of the second order (equations (19), below). To this end we now make use of (13) and insert into it the functions $g(x, y, z)$ and $f(x, y, z)$, given respectively by equations (17) and (11a), and we arrange in powers of the independent variable z. As a result, we obtain, after some straightforward algebra, an algebraic equation of the third degree in z. The four coefficients are functions of x, y only and must be identically equal to zero.

The fact that the coefficient of z^3 is zero leads to the equation

$$h_x + 4AU_{10} + BU_{01} = 0. \tag{18}$$

On the other hand, the system (call it: system (S_2)) of the three equations, obtained from the conditions that the coefficients of z^2, z and z^0 must be equal to zero identically, include linearly first and second-order derivatives with respect to x, y of the function $I(x, y)$. Solving the system (S_2) for I_{xx}, I_{xy}, I_{yy} and taking into account (18), we obtain

$$I_{xx} = \frac{U_{11}}{U_{01}} I_x + U_{01} h_y + \frac{2}{U_{01}} (U_{10}U_{11} - U_{01}U_{20}) h$$
$$+ (8AU_{10}^2 + 5BU_{01}U_{10} + 2CU_{01}^2) \tag{19a}$$

$$I_{xy} = \mu_1 I_x + \frac{U_{11}}{3U_{01}} I_y - U_{10} h_y + \mu_2 h + (BU_{10}^2 + 2CU_{01}U_{10} + 2DU_{01}^2) \tag{19b}$$

$$I_{yy} = \mu_3 I_x + \mu_4 I_y + \frac{U_{10}^2}{U_{01}} h_y + \mu_5 h$$
$$+ \frac{1}{U_{01}} (BU_{10}^3 + 2CU_{01}U_{10}^2 + 4DU_{01}^2 U_{10} + 8EU_{01}^3) \tag{19c}$$

where we introduced the notation

$$\mu_1=\frac{2U_{01}U_{02}-U_{10}U_{11}+U_{01}U_{20}}{3U_{01}^2} \tag{20a}$$

$$\mu_2=\frac{2U_{10}}{3U_{01}}(2U_{02}+U_{20})-\tfrac{2}{3}U_{11}\left(2-\frac{U_{10}^2}{U_{01}^2}\right) \tag{20b}$$

$$\mu_3=\frac{U_{01}U_{10}(U_{20}-4U_{02})+(3U_{01}^2-U_{10}^2)U_{11}}{3U_{01}^3} \tag{20c}$$

$$\mu_4=\frac{U_{10}U_{11}+3U_{01}U_{02}}{3U_{01}^2} \tag{20d}$$

$$\mu_5=\frac{8U_{10}}{3U_{01}^2}(U_{01}U_{11}-U_{02}U_{10})+\frac{2U_{10}^2}{3U_{01}^3}(U_{01}U_{20}-U_{10}U_{11}). \tag{20e}$$

Working out the compatibility conditions for equations (19) and replacing second-order derivatives in terms of first-order derivatives, we obtain, as expected, an algebraic system of two, linear in I_x, I_y, equations (call it system (S_1)).

Recall that, up to now, for a given $U(x, y)$, function $h(x, y)$ is only subject to the condition (18).

The calculations which follow become pretty tedious, yet straightforward. They were done by a REDUCE program with the IBM 4381/M13 computer of the University of Thessaloniki. Before writing down the system (S_1), (which is essentially the system of equations (21) below), we offer sufficient explanations regarding the algebra:

(i) Making use of (15) we replaced the partial derivatives with respect to y of the coefficients A, B, C, D by their corresponding x-derivatives of B, C, D, E.

(ii) First and second-order derivatives of $h(x, y)$, entering into (S_1), were expressed in terms of h_y and h_{yy}; in other words: we did not allow for derivatives h_x, h_{xx}, h_{xy} to enter into the calculations. The purpose of this practice will become clear in what follows.

Assuming that $J\neq 0$ and observing the above prescriptions we solved the system (S_1) for I_x, I_y and we obtained

$$I_x=J^{-1}(K_2h_{yy}+K_1h_y+K_0h+K) \tag{21a}$$

$$I_y=J^{-1}(L_2h_{yy}+L_1h_y+L_0h+L) \tag{21b}$$

with

$$\begin{aligned}J=3U_{01}U_{12}U_{30}-4U_{02}U_{11}U_{30}+U_{11}U_{20}U_{30}+3U_{01}U_{03}U_{21}-5U_{03}U_{11}^2-3U_{01}U_{12}^2\\+9U_{02}U_{11}U_{12}-6U_{11}U_{12}U_{20}-3U_{01}U_{21}^2-4U_{02}^2U_{21}+5U_{02}U_{20}U_{21}\\+5U_{11}^2U_{21}-U_{20}^2U_{21}\end{aligned} \tag{22}$$

$$K_2=3U_{01}(3U_{12}U_{01}^2+3U_{01}U_{10}U_{21}-4U_{01}U_{02}U_{11}-5U_{10}U_{11}^2+U_{01}U_{11}U_{20}) \tag{23a}$$

$$\begin{aligned}K_1=U_{01}(3U_{01}U_{02}U_{12}+15U_{10}U_{11}U_{12}-12U_{01}U_{12}U_{20}-12U_{02}U_{10}U_{21}+15U_{01}U_{11}U_{21}\\+3U_{10}U_{20}U_{21}-4U_{02}^2U_{11}+17U_{02}U_{11}U_{20}-25U_{11}^3-4U_{11}U_{20}^2)\end{aligned} \tag{23b}$$

$$K_0=-2U_{10}J \tag{23c}$$

$$K = K_{0a}A + K_{0b}B + K_{0c}C + K_{0d}D + K_{0e}E + K_{1b}B_x + K_{1c}C_x + K_{1d}D_x + K_{1e}E_x \tag{23d}$$

where

$$K_{0a} = 20\,U_{01}U_{10}U_{11}(3\,U_{01}U_{12} - 4\,U_{02}U_{11} + U_{11}U_{20}) \tag{23e}$$

$$\begin{aligned} K_{0b} = {} & U_{01}^2(18\,U_{02}U_{10}U_{12} + 33\,U_{01}U_{11}U_{12} - 27\,U_{10}U_{12}U_{20} + 18\,U_{10}U_{11}U_{21} - 44\,U_{02}U_{11}^2 \\ & + 11\,U_{11}^2U_{20}) + 3\,U_{01}U_{10}(5\,U_{10}U_{11}U_{12} - 4\,U_{02}U_{10}U_{21} + U_{10}U_{20}U_{21} \\ & - 8\,U_{02}^2U_{11} + 14\,U_{02}U_{11}U_{20} - 10\,U_{11}^3 - 3\,U_{11}U_{20}^2) \end{aligned} \tag{23f}$$

$$\begin{aligned} K_{0c} = {} & 2U_{01}^2(12\,U_{01}U_{02}U_{12} - 12\,U_{01}U_{12}U_{20} - 3\,U_{02}U_{10}U_{21} + 3\,U_{10}U_{20}U_{21} - 16\,U_{02}^2U_{11} \\ & + 20\,U_{02}U_{11}U_{20} - 4\,U_{11}U_{20}^2) + 10\,U_{01}U_{10}U_{11}^2(U_{02} - U_{20}) \end{aligned} \tag{23g}$$

$$\begin{aligned} K_{0d} = {} & 3\,U_{01}^2(-8\,U_{01}U_{11}U_{12} - 5\,U_{01}U_{02}U_{21} + 2\,U_{10}U_{11}U_{21} + 5\,U_{01}U_{20}U_{21} + 19\,U_{02}U_{11}^2 \\ & - 11\,U_{20}U_{11}^2) - 10\,U_{01}U_{10}U_{11}^3 \end{aligned} \tag{23h}$$

$$K_{0e} = 20\,U_{01}^2U_{11}(3\,U_{01}U_{21} - 5\,U_{11}^2) \tag{23i}$$

$$K_{1b} = 15\,U_{01}U_{10}^2(-3\,U_{01}U_{12} + 4\,U_{02}U_{11} - U_{11}U_{20}) \tag{23j}$$

$$K_{1c} = 9U_{01}^2U_{10}(-6\,U_{01}U_{12} - U_{10}U_{21} + 8\,U_{02}U_{11} - 2\,U_{11}U_{20}) + 15\,U_{01}U_{10}^2U_{11}^2 \tag{23k}$$

$$K_{1d} = 3\,U_{01}^3(-12\,U_{01}U_{12} + 3\,U_{10}U_{21} + 16\,U_{02}U_{11} - 4\,U_{11}U_{20}) - 15\,U_{01}^2U_{10}U_{11}^2 \tag{23l}$$

$$K_{1e} = 15\,U_{01}^3(3\,U_{01}U_{21} - 5\,U_{11}^2) \tag{23m}$$

and

$$\begin{aligned} L_2 = {} & -15\,U_{10}^2U_{11}^2 + 3\,U_{01}(-3\,U_{01}U_{10}U_{30} + 3\,U_{01}^2U_{03} + 6\,U_{01}U_{10}U_{12} + 3\,U_{10}^2U_{21} - 3\,U_{01}^2U_{21} \\ & - 4\,U_{01}U_{02}^2 - 9\,U_{02}U_{10}U_{11} + 5\,U_{01}U_{02}U_{20} + 6\,U_{10}U_{11}U_{20} - U_{01}U_{20}^2) \end{aligned} \tag{24a}$$

$$\begin{aligned} L_1 = {} & U_{01}(12\,U_{02}U_{10}U_{30} - 12\,U_{01}U_{03}U_{20} + 12\,U_{01}U_{20}U_{21} - 15\,U_{01}U_{11}U_{30} + 15\,U_{03}U_{10}U_{11} \\ & + 15\,U_{01}U_{11}U_{12} - 3\,U_{10}U_{20}U_{30} + 3\,U_{01}U_{02}U_{03} - 3\,U_{01}U_{02}U_{21} - 9\,U_{02}U_{10}U_{12} \\ & - 9\,U_{10}U_{12}U_{20} + 21\,U_{02}^2U_{20} - 21\,U_{02}U_{20}^2 - 4\,U_{02}^3 + 4\,U_{20}^3 - 25\,U_{02}U_{11}^2 \\ & + 25\,U_{11}^2U_{20}) + U_{10}(15\,U_{10}U_{11}U_{12} - 12\,U_{02}U_{10}U_{21} \\ & + 3\,U_{10}U_{20}U_{21} - 4\,U_{02}^2U_{11} + 17\,U_{02}U_{11}U_{20} - 25\,U_{11}^3 - 4\,U_{11}U_{20}^2) \end{aligned} \tag{24b}$$

$$L_0 = -2\,U_{01}J \tag{24c}$$

$$L = L_{0a}A + L_{0b}B + L_{0c}C + L_{0d}D + L_{0e}E + L_{1b}B_x + L_{1c}C_x + L_{1d}D_x + L_{1e}E_x \tag{24d}$$

where

$$L_{0a}=20U_{01}U_{10}U_{11}(3U_{01}U_{03}+3U_{10}U_{12}-3U_{01}U_{21}-4U_{02}^2+5U_{02}U_{20}-U_{20}^2)$$
$$+20U_{10}^2U_{11}^2(U_{20}-4U_{02}) \qquad (24e)$$

$$L_{0b}=U_{01}^2(-18U_{10}U_{11}U_{30}+18U_{02}U_{03}U_{10}+33U_{01}U_{03}U_{11}-27U_{03}U_{10}U_{20}$$
$$+51U_{10}U_{11}U_{12}-18U_{02}U_{10}U_{21}-33U_{01}U_{11}U_{21}+27U_{10}U_{20}U_{21}$$
$$-44U_{02}^2U_{11}+55U_{02}U_{11}U_{20}-11U_{11}U_{20}^2)$$
$$+U_{01}U_{10}(12U_{02}U_{10}U_{30}-3U_{10}U_{20}U_{30}+15U_{03}U_{10}U_{11}+6U_{02}U_{10}U_{12}$$
$$-24U_{10}U_{12}U_{20}+3U_{10}U_{11}U_{21}-24U_{02}^3+66U_{02}^2U_{20}-74U_{02}U_{11}^2$$
$$-51U_{02}U_{20}^2+41U_{11}^2U_{20}+9U_{20}^3)$$
$$+3U_{10}^2(5U_{10}U_{11}U_{12}-4U_{02}U_{10}U_{21}+U_{10}U_{20}U_{21}-8U_{02}^2U_{11}$$
$$+14U_{02}U_{11}U_{20}-10U_{11}^3-3U_{11}U_{20}^2) \qquad (24f)$$

$$L_{0c}=2U_{01}^2(3U_{02}U_{10}U_{30}-3U_{10}U_{20}U_{30}+12U_{01}U_{02}U_{03}-12U_{01}U_{03}U_{20}+9U_{02}U_{10}U_{12}$$
$$-9U_{10}U_{12}U_{20}-12U_{01}U_{02}U_{21}+12U_{01}U_{20}U_{21}-16U_{02}^3+36U_{02}^2U_{20}$$
$$-24U_{02}U_{20}^2+4U_{20}^3)+2U_{01}U_{10}U_{11}(-11U_{02}^2+10U_{02}U_{20}+U_{20}^2)$$
$$+2U_{10}^2(-3U_{01}U_{02}U_{21}+3U_{01}U_{20}U_{21}+5U_{02}U_{11}^2-5U_{11}^2U_{20}) \qquad (24g)$$

$$L_{0d}=3U_{01}^2(5U_{01}U_{02}U_{30}-2U_{10}U_{11}U_{30}-5U_{01}U_{20}U_{30}-8U_{01}U_{03}U_{11}-5U_{01}U_{02}U_{12}$$
$$-6U_{10}U_{11}U_{12}+5U_{01}U_{12}U_{20}-5U_{02}U_{10}U_{21}+8U_{01}U_{11}U_{21}+5U_{10}U_{20}U_{21}$$
$$+19U_{02}^2U_{11}-30U_{02}U_{11}U_{20}+11U_{11}U_{20}^2)$$
$$+U_{01}U_{10}U_{11}(6U_{10}U_{21}+47U_{02}U_{11}-23U_{11}U_{20})-10U_{10}^2U_{11}^3 \qquad (24h)$$

$$L_{0e}=20U_{01}^2U_{11}(-3U_{01}U_{30}+3U_{01}U_{12}+3U_{10}U_{21}-5U_{02}U_{11}+5U_{11}U_{20})$$
$$-100U_{01}U_{10}U_{11}^3 \qquad (24i)$$

$$L_{1b}=15U_{10}^2(-3U_{01}^2U_{03}-3U_{01}U_{10}U_{12}+3U_{01}^2U_{21}+4U_{01}U_{02}^2+4U_{02}U_{10}U_{11}$$
$$-5U_{01}U_{02}U_{20}-U_{10}U_{11}U_{20}+U_{01}U_{20}^2) \qquad (24j)$$

$$L_{1c}=9U_{01}^2U_{10}(U_{10}U_{30}-6U_{01}U_{03}-7U_{10}U_{12}+6U_{01}U_{21}+8U_{02}^2-10U_{02}U_{20}+2U_{20}^2)$$
$$+3U_{01}U_{10}^2(29U_{02}U_{11}-11U_{11}U_{20})+3U_{10}^3(5U_{11}^2-3U_{01}U_{21}) \qquad (24k)$$

$$L_{1d}=3U_{01}^3(-3U_{10}U_{30}-12U_{01}U_{03}-9U_{10}U_{12}+12U_{01}U_{21}+16U_{02}^2-20U_{02}U_{20}+4U_{20}^2)$$
$$+3U_{01}^2U_{10}(3U_{10}U_{21}+11U_{02}U_{11}+U_{11}U_{20})-15U_{01}U_{10}^2U_{11}^2 \qquad (24l)$$

$$L_{1e}=15U_{01}^3(-3U_{01}U_{30}+3U_{01}U_{12}+3U_{10}U_{21}-5U_{02}U_{11}+5U_{11}U_{20})$$
$$-75U_{01}^2U_{10}U_{11}^2. \qquad (24m)$$

From now on, for any given potential function $U(x, y)$, apart from the function J, we shall consider as known functions all the Ks and Ls subscripted by a number or by a number and a letter.

The aim of the present section is to write down the conditions for the existence of the function $I(x, y)$. To this end, certain new functions, denoted by M subscripted by a number or by a pair of a number and a letter will be defined. They will be expressed in terms of J and the 26 functions K and L and derivatives of these.

Suppose then for the moment that, apart from $U(x, y)$, we were given 'appropriate' values of the 15 constants (16*) as well as an 'arbitrary function' $h(x, y)$ (satisfying, of course, equation (18)), so that the system (21) admitted of a solution which also was in agreement with all equations (19). In this case, $U(x, y)$ would be integrable and the corresponding constant of motion could be found. Indeed, function $g(x, y, z)$ would be known from equation (17). The three coefficients F, G, H could be found immediately from equation (11b), the coefficients A, B, C, D, E would be known from (16), so the integral (9) could be written down.

So we focus our attention on the system (21) from which we expect to complete our information regarding the arbitrary function $h(x, y)$. The necessary and sufficient condition for this system to admit of a solution leads to the condition

$$M_3 h_{yyy} + M_2 h_{yy} + M_1 h_y = M \tag{25}$$

where

$$M = L_x J - L J_x + J_y K - J K_y + J(L_2 h_{xyy} + L_1 h_{xy} + L_0 h_x) \tag{26}$$

and M_3, M_2, M_1 are given in the appendix.

In what follows we shall use successively the functions

$$M, N, P, Q \text{ and } R^{(k)} \qquad (k = 1, 2, 3, 4) \tag{27}$$

all having numbers or numbers and letters as indices. They are all given in turn in the appendix.

Comments. (i) The coefficients M_3, M_2, M_1 are not given explicitly in terms of U_{ij} but in terms of the basic functions J; K_2, K_1, L_2, L_1 and partial derivatives of these. It is easy to see, however, that these coefficients are again homogeneous polynomials in U_{ij}, including derivatives of $U(x, y)$ up to fourth order.

(ii) In applying the integrability condition to the equations (21) one would normally expect a term $M_0 h$ to be present in the left-hand side of (25). However it so happens (and this is easy to show) that the coefficient

$$M_0 = (K_{0y} - L_{0x})J + L_0 J_x - K_0 J_y$$

of h in (25) is identically equal to zero. This is, of course, a happy coincidence, simplifying considerably the calculations which follow.

We now write equation (26) as follows

$$M = M_{0a}A + M_{0b}B + M_{0c}C + M_{0d}D + M_{0e}E + M_{1b}B_x + M_{1c}C_x + M_{1d}D_x + M_{1e}E_x + M_{2c}C_{xx} + M_{2d}D_{xx} + M_{2e}E_{xx}. \tag{28}$$

The coefficients $M_{0a}, \ldots, M_{2e}$ are given in the appendix. The above equation is a typical expansion of M in terms of the polynomials (16) and derivatives of these with respect to x. We have already used such expansions for K and L in (23d) and (24d). In the sequel we shall be using such typical expansions of the functions N, P, Q also. It is only at the end of the analysis that we shall introduce, instead of the polynomials (16), the 15 constants (16*) of the problem.

In conclusion, we proved in this section that $I(x, y)$ needed for the construction of the constant (9), must be found from system (21), provided that the arbitrary function $h(x, y)$ satisfies, for a given $U(x, y)$, conditions (25) and (18). Not to be forgotten, however, is that appropriate functions $I(x, y)$ and $h(x, y)$ satisfying all equations (19) should exist so that the system is integrable. The question then is: Is the solution $I(x, y)$ of (21) good for this purpose? For an affirmative answer three additional conditions should be imposed on $h(x, y)$.

5. A system of equations for the function $h(x, y)$

We now insert I_x, I_y, given by (21), into equation (19a) and, making use of (18), we obtain

$$N_2 h_{yy} + N_1 h_y = N \tag{29}$$

where N is expanded like M in (28) and the functions N_2, N_1 as well as $N_{0a}, \ldots, N_{2e}$ are given in the appendix.

Next we insert (21) into (19b). Again taking into account (18) we find

$$P_2 h_{yy} + P_1 h_y = P \tag{30}$$

where P has a typical expansion similar to (28) and the functions P_2, P_1; $P_{0a}, \ldots, P_{2e}$ are given in the appendix. Finally we insert (21) into (19c) and obtain

$$Q_3 h_{yyy} + Q_2 h_{yy} + Q_1 h_y = Q \tag{31}$$

with an expansion of Q of the form (28) and Q_3, Q_2, Q_1; $Q_{0a}, \ldots, Q_{2e}$, all given in the appendix.

Thus the arbitrary function $h(x, y)$ has to satisfy five equations which we now bring together

$$h_x = -4AU_{10} - BU_{01} \tag{32a}$$

$$N_2 h_{yy} + N_1 h_y = N \tag{32b}$$

$$P_2 h_{yy} + P_1 h_y = P \tag{32c}$$

$$M_3 h_{yyy} + M_2 h_{yy} + M_1 h_y = M \tag{32d}$$

$$Q_3 h_{yyy} + Q_2 h_{yy} + Q_1 h_y = Q. \tag{32e}$$

If, for a given $U(x, y)$, we can find 15 constants (16*), not all zero, such that the system (32) is compatible for $h = h(x, y)$, then $U(x, y)$ is integrable, admitting a quartic of the form (9) which can be constructed by finding successively the functions $h(x, y)$, $I(x, y)$ and $g(x, y, z)$, the existence of which is guaranteed.

There are several ways of facing a problem like the above, i.e. finding conditions on U_{ij} under which the system (32) is compatible. In fact, it may be that various treatments lead to different-looking sets of conditions, at least at first sight. We shall write down four conditions on U_{ij} and give a formula for h_y in terms of U_{ij}. This formula for h_y, combined with (32a), then serves to determine $h(x, y)$ up to an additive constant.

6. Compatibility conditions for the existence of $h(x, y)$

The necessary and sufficient conditions (equations (38) below, for $k=1,2,3,4$) for the system (32) to be compatible are obtained by assuming that certain denominators appearing in the analysis are not zero. The special cases of having any of these denominators equal to zero are commented upon at the end of this section.

Let us then introduce

$$\delta = P_2N_1 - P_1N_2 \tag{33}$$

assume that

$$\delta \neq 0 \tag{34}$$

and solve (32*b*), (32*c*) for h_y, h_{yy}.

Assuming further that

$$M_3 \neq 0 \qquad Q_3 \neq 0 \tag{35}$$

we rewrite the system (32) as follows

$$h_x = -4AU_{10} - BU_{01} \tag{36a}$$

$$h_y = \frac{1}{\delta}(P_2N - N_2P) \tag{36b}$$

$$h_{yy} = \frac{1}{\delta}(N_1P - P_1N) \tag{36c}$$

$$h_{yyy} = \frac{1}{\delta M_3}[\delta M + (M_1N_2 - M_2N_1)P + (M_2P_1 - M_1P_2)N] \tag{36d}$$

$$h_{yyy} = \frac{1}{\delta Q_3}[\delta Q + (Q_1N_2 - Q_2N_1)P + (Q_2P_1 - Q_1P_2)N]. \tag{36e}$$

We seek the necessary and sufficient conditions so that the above system (36), which is equivalent to (32), is compatible. Among the various ways of establishing this compatibility we proceed in four steps by making compatible

(i) (36*a*) and (36*b*) (ii) (36*b*) and (36*c*)

(iii) (36*c*) and (36*d*) (iv) (36*d*) and (36*e*)

As we work out each of the above four steps we obtain four conditions for the potential function $U(x, y)$, including only the 15 constants (16*). These U-conditions are in turn (for $k=1,2,3,4$) of the typical form

$$\begin{aligned} R^{(k)}_{0a}A + R^{(k)}_{0b}B + R^{(k)}_{0c}C + R^{(k)}_{0d}D + R^{(k)}_{0e}E + R^{(k)}_{1b}B_x + R^{(k)}_{1c}C_x \\ + R^{(k)}_{1d}D_x + R^{(k)}_{1e}E_x + R^{(k)}_{2c}C_{xx} + R^{(k)}_{2d}D_{xx} + R^{(k)}_{2e}E_{xx} \\ + R^{(k)}_{3d}D_{xxx} + R^{(k)}_{3e}E_{xxx} = 0 \end{aligned} \tag{37}$$

where all the coefficients $R^{(k)}$ $(k=1,2,3,4)$ are given in the appendix. If adequate polynomials (16) exist such that the four conditions (37) are satisfied and if the inequalities (34) and (35) are satisfied then the system (32) does have a solution which can be found from (32*a*) and (32*b*). The additive constant appearing in $h(x, y)$ may be taken equal to zero as explained in section 8.

Comment. If $\delta=0$, then, for the system of (32*b*), (32*c*) to admit of a solution, we must have $P_1N-N_1P=0$ which is itself a U-condition. Thus, at the expense of disregarding one of the above two equations, we gain a U-condition of the form (37). There will be another three conditions from the remaining equations (32). If $P_1N-N_1P=0$ happens to be an identity then, of course, (32*b*) and (32*c*) are identical and the U-conditions are reduced to three.

The same reasoning is good for $M_3=0$ or $Q_3=0$. We shall give no additional details for these special cases, which can be studied separately if they arise. It is clear, however, that there is no way to know in advance if functions $U(x,y)$ satisfying, for instance, $M_3=0$ are integrable with a quartic integral.

7. Integrability conditions for the function $U(x,y)$

At a final stage, in view of (16), we write the four conditions (37) for $k=1,2,3,4$, found in section 6 as linear expressions of the 15 constants (16*) with coefficients which depend merely on the derivatives U_{ij}, with $i+j\leqslant 5$. We then obtain four expressions of the form:

$$A_4^{(k)}a_4+A_3^{(k)}a_3+A_2^{(k)}a_2+A_1^{(k)}a_1+A_0^{(k)}a_0+B_3^{(k)}b_3+B_2^{(k)}b_2+B_1^{(k)}b_1+B_0^{(k)}b_0+C_2^{(k)}c_2 \\ +C_1^{(k)}c_1+C_0^{(k)}c_0+D_1^{(k)}d_1+D_0^{(k)}d_0+E_0^{(k)}e_0=0 \tag{38}$$

with $k=1,2,3,4$. The coefficients are

$$A_4^{(k)}=(y^4R_{0a}^{(k)}-4xy^3R_{0b}^{(k)}+6x^2y^2R_{0c}^{(k)}-4x^3yR_{0d}^{(k)}+x^4R_{0e}^{(k)}) \\ -4(y^3R_{1b}^{(k)}-3xy^2R_{1c}^{(k)}+3x^2yR_{1d}^{(k)}-x^3R_{1e}^{(k)}) \\ +12(y^2R_{2c}^{(k)}-2xyR_{2d}^{(k)}+x^2R_{2e}^{(k)})-24(yR_{3d}^{(k)}-xR_{3e}^{(k)}) \tag{39a}$$

$$A_3^{(k)}=(y^3R_{0a}^{(k)}-3xy^2R_{0b}^{(k)}+3x^2yR_{0c}^{(k)}-x^3R_{0d}^{(k)})-3(y^2R_{1b}^{(k)}-2xyR_{1c}^{(k)}+x^2R_{1d}^{(k)}) \\ +6(yR_{2c}^{(k)}-xR_{2d}^{(k)})-6R_{3d}^{(k)} \tag{39b}$$

$$A_2^{(k)}=(y^2R_{0a}^{(k)}-2xyR_{0b}^{(k)}+x^2R_{0c}^{(k)})-2(yR_{1b}^{(k)}-xR_{1c}^{(k)})+2R_{2c}^{(k)} \tag{39c}$$

$$A_1^{(k)}=(yR_{0a}^{(k)}-xR_{0b}^{(k)})-R_{1b}^{(k)} \tag{39d}$$

$$A_0^{(k)}=R_{0a}^{(k)} \tag{39e}$$

$$B_3^{(k)}=(y^3R_{0b}^{(k)}-3xy^2R_{0c}^{(k)}+3x^2yR_{0d}^{(k)}-x^3R_{0e}^{(k)})-3(y^2R_{1c}^{(k)}-2xyR_{1d}^{(k)}+x^2R_{1e}^{(k)}) \\ +6(yR_{2d}^{(k)}-xR_{2e}^{(k)})-6R_{3e}^{(k)} \tag{39f}$$

$$B_2^{(k)}=(y^2R_{0b}^{(k)}-2xyR_{0c}^{(k)}+x^2R_{0d}^{(k)})-2(yR_{1c}^{(k)}-xR_{1d}^{(k)})+2R_{2d}^{(k)} \tag{39g}$$

$$B_1^{(k)}=(yR_{0b}^{(k)}-xR_{0c}^{(k)})-R_{1c}^{(k)} \tag{39h}$$

$$B_0^{(k)}=R_{0b}^{(k)} \tag{39i}$$

$$C_2^{(k)}=(y^2R_{0c}^{(k)}-2xyR_{0d}^{(k)}+x^2R_{0e}^{(k)})-2(yR_{1d}^{(k)}-xR_{1e}^{(k)})+R_{2e}^{(k)} \tag{39j}$$

$$C_1^{(k)} = (yR_{0c}^{(k)} - xR_{0d}^{(k)}) - R_{1d}^{(k)} \tag{39k}$$

$$C_0^{(k)} = R_{0c}^{(k)} \tag{39l}$$

$$D_1^{(k)} = (yR_{0d}^{(k)} - xR_{0e}^{(k)}) - R_{1e}^{(k)} \tag{39m}$$

$$D_0^{(k)} = R_{0d}^{(k)} \tag{39n}$$

$$E_0^{(k)} = R_{0e}^{(k)}. \tag{39o}$$

Equations (38) for $k=1,2,3,4$ constitute our final result. They include the 15 constants (16*) and derivatives U_{ij} $(i+j \leq 5)$ with respect to x, y of the given potential function $U(x, y)$, up to the fifth order. They have been derived and they stand for the necessary and sufficient conditions for the existence of the arbitrary function $h(x, y)$ which first appeared in the general solution (17) of the differential equation (14). Not to be forgotten, of course, is that the four equations (37) have been derived for

$$J \neq 0 \qquad \delta \neq 0 \qquad M_3 \neq 0 \qquad Q_3 \neq 0. \tag{40}$$

The case $J=0$ is discussed in section 10. For $J \neq 0$, even if one of δ, M_3, Q_3 vanishes there can be found four conditions of the form (38). Equations (39) will still be valid but the functions $R_{0a}^{(k)} \ldots R_{3e}^{(k)}$ for $k=1,2,3,4$ are no longer those given in the appendix but they have to be found according to the specific circumstance.

Thus we state our final conclusion.

Theorem. If (apart from the exception discussed in section 9), for a given potential function U satisfying the inequalities (40), there exist 15 constants (16*) *not all zero,* such that the four equations (38) are satisfied for $k=1,2,3,4$, then the potential function U is integrable.

Apart from certain exceptional cases which correspond to the existence of the first-order integral of angular momentum (not to mention of course the quite degenerate cases accounting for constant momentum along a certain direction) and which are discussed in section 9, the integral is a genuine fourth-order algebraic constant of motion and it can be constructed by tracing back the steps of the analysis presented in this paper, as explained in the next section.

8. Summary of the algorithm—construction of the integral

To test for integrability, of the sort discussed in this paper, a given potential function $U(x, y)$ we proceed as follows:

(i) We calculate J from (22) and all the functions K and L from (23) and (24).

(ii) We prepare the functions μ_1, μ_3, μ_4 given by (20a), (20c) and (20d). (The functions μ_2 and μ_5 are not needed; they were factoring h in (19b) and (19c) and h itself does not appear in the typical expansions.)

(iii) From the appendix we calculate all the functions M, N, P, Q.

(iv) We no longer need J and the functions K and L. In terms of M, N, P, Q we prepare all the functions $R^{(k)}$ for $k=1,2,3,4$. They are all given in the appendix. The functions $\delta_1, \delta_2, \delta_3, \delta_4$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, needed for the calculation of the $R^{(k)}$s, are given in the proper place. The function δ is given by (33).

(v) Finally we find from equations (39), for $k = 1, 2, 3, 4$, the 60 functions $A_4^{(k)}, A_3^{(k)}, \ldots, E_0^{(k)}$ and apply the theorem of section 7. It is an easy task to find the 15 constants $a_4, a_3, \ldots, e_0$ *for which each condition* (38) *is satisfied or to conclude* that such constants (not all zero) do not exist.

Exceptional cases are to be treated accordingly, as described in section 6 ($\delta = 0$, $M_3 = 0$, $Q_3 = 0$) and in section 10 ($J = 0$).

Having fixed the values of the constants we proceed to the construction of the integral of motion.

(vi) We find, from (36*a*), (36*b*), the function

$$h(x, y) = h_0(x, y) + c. \tag{41}$$

It can be shown that the additive constant c in (41) results in an extra term $c(\dot{x}^2 + \dot{y}^2) - 2cU = 2cE$ to be added to the integral (9). So we put $c = 0$.

(vii) Using $h = h_0(x, y)$ we determine $I(x, y)$ from (21), up to an additive constant, which again may be put equal to zero.

(viii) Finally, the coefficients F, G, H in (9) are found from (17) and (11*b*). They are

$$F = h_0(x, y) \tag{42a}$$

$$G = -\frac{1}{U_{01}}(2U_{10}h_0 + I_x) \tag{42b}$$

$$H = \frac{1}{2U_{01}^2}(2U_{10}^2h_0 + U_{10}I_x - U_{01}I_y). \tag{42c}$$

9. Degenerate cases—some identities

We mentioned already that the theorem stated at the end of section 7 is valid, apart from an exception. The obvious exception refers to the case of taking as 'second' integral the expression

$$\varphi = \dot{x}^4 + 2\dot{x}^2\dot{y}^2 + \dot{y}^4 - 4U\dot{x}^2 - 4U\dot{y}^2 + 4U^2 \tag{43}$$

i.e. the square of the constant $2E = \dot{x}^2 + \dot{y}^2 - 2U$, given by equation (2). This corresponds to $A = 1$, $B = 0$, $C = 2$, $D = 0$, $E = 1$ or, in terms of the constants (16*), to $a_0 = 1$, $c_0 = 2$, $e_0 = 1$ (or multiples of this triplet) and all the 12 other constants equal to zero.

It can be shown by direct calculations, that $A_0^{(k)} + 2C_0^{(k)} + E_0^{(k)}$ is identically equal to zero, or that

$$R_{0a}^{(k)} + 2R_{0c}^{(k)} + R_{0e}^{(k)} = 0 \qquad (k = 1, 2, 3, 4) \tag{44}$$

for all potentials but this, of course, does not imply that the given potential is integrable.

Apart from (44), other identities *relating the functions used in this paper can be* obtained. Thus, for instance, we can identify (43) with (9) for $g(x, y, z) = -4Uz^2 - 4U$. In view of (17) we then have $h = -4U$ and, from equations (21), we obtain the identities

$$4U_{02}K_2 + 4U_{01}K_1 = K_{0a} + 2K_{0c} + K_{0e} \tag{45a}$$

$$4U_{02}L_2 + 4U_{01}L_1 = L_{0a} + 2L_{0c} + L_{0e}. \tag{45b}$$

The theorem of section 7 is also applicable for potentials associated with integrals of the first or of the second-degree in the velocity components. The integrals so detected are then pseudo-quartics.

We examine now very briefly two degenerate cases, both referring to a central field

$$U = U(r) \qquad r = (x^2+y^2)^{1/2}. \tag{46}$$

(i) Since the angular momentum $C = \dot{x}y - x\dot{y}$ is constant, $\varphi = (\dot{x}y - x\dot{y})^4$ is also an integral of motion corresponding to $a_4 = 1$ and all the remaining 14 constants (16*) equal to zero. Then

$$A_4^{(k)} = 0 \tag{47}$$

for $k = 1, 2, 3, 4$ and this can be checked to be true for all potentials of the form (46).

So for $a_4 = 1$ and the remaining constants zero the theorem asserts the existence of the angular momentum, not of a genuine quartic. Could it be that for non-central potentials we find genuine quartics with $a_4 = 1$ and zero all the other constants? The answer to this question is not affirmative and this was shown by Leach [11] also.

(ii) The 'quartic' $\varphi = 2EC^2$, corresponds to $a_2 = 1, c_2 = 1$ and zero the other 13 constants (16*). From (42) we obtain

$$A_2^{(k)} + C_2^{(k)} = 0 \tag{48}$$

for all potentials of the form (46).

In the same manner one could apply the theorem for other pseudo-quartics like $\varphi = C^4 + C^2$, $\varphi = 2EC^2 + C^4$, etc. but, of course, there is nothing more to it except for checking the correctness of the basic functions K and L of this paper.

10. The case $J = 0$

Up to now we have assumed that $J \neq 0$. Some special cases, commented at the end of section 6, did not call for essential alteration of the method. However, if $J = 0$, the method developed in this paper does not work. Since for $J = 0$ both K_0 and L_0 vanish, from equations (21) it is seen that we must have in this case

$$K_2 h_{yy} + K_1 h_y + K = 0 \tag{49a}$$

$$L_2 h_{yy} + L_1 h_y + L = 0 \tag{49b}$$

but I_x, I_y become indeterminate.

To examine this case separately we define

$$\delta^* = L_2 K_1 - L_1 K_2 \tag{50}$$

and distinguish two subcases:

Case (i)

$$\delta^* \neq 0. \tag{51}$$

The arbitrary function $h(x, y)$ has to satisfy the following conditions:

$$h_x = -4AU_{10} - BU_{01} \tag{52a}$$

$$h_y = \frac{1}{\delta^*}(K_2 L - L_2 K) \tag{52b}$$

$$h_{yy} = \frac{1}{\delta^*}(L_1 K - K_1 L). \tag{52c}$$

The system (52) is of the same type as the system of the three equations (36*a*), (36*b*), (36*c*). Exact correspondence is established if we set

$$\begin{array}{lll} N_2 \to K_2 & N_1 \to K_1 & N \to -K \\ P_2 \to L_2 & P_1 \to L_1 & P \to -L \end{array} \tag{53a}$$

in which case

$$\delta \to \delta^*. \tag{53b}$$

To make compatible the system (52) we repeat the first two steps (i) and (ii) described in section 6. We then obtain two conditions of the form (37) for $k=1,2$. Therefore the theorem of section 7 must be checked only for $k=1$ and $k=2$. The coefficients $A_4^{(k)}, \ldots, E_0^{(k)}$ $(k=1,2)$ are again given by (39) and the functions $R_{0a}^{(k)} \ldots R_{3e}^{(k)}$ are to be taken from the appendix for $k=1,2$, with the provision of the correspondence established by (53).

Case (ii)

$$\delta^* = 0. \tag{54}$$

Then

$$K_2 L - L_2 K = 0 \tag{55}$$

and the two equations (49) are identical. We find $h(x, y)$ from (52*a*) and one of equations (49), say (49*a*). As to the function $I(x, y)$, it will be found from (19). In fact the second example in section 11 falls into this case and will be treated in some detail.

11. Examples

We shall apply the theorem of section 7 in three examples: the first, for $J \neq 0$, leads to a genuine quartic integral of motion; the second, for $J=0$, leads to a quadratic integral; the third stands for a counterexample. The calculations were performed by a REDUCE program, available in the IBM 4381/M13 computer of the University of Thessaloniki:

(a) The integrable potential function

$$U = -\tfrac{16}{3}x^3 - xy^2 \tag{56}$$

is given by Hiertarinta [4]. To establish the integrability of (56) with the aid of the theorem of this paper we follow the steps (i)–(v) summarized in section 8. Since

$$J = -1200xy \neq 0$$

$$\delta = 82944 \times 10^7\, x^5 y^6 (546x^2 - 29y^2) \neq 0$$

$$M_3 = 144 \times 10^3\, x^2 y^4 (10x^2 + y^2) \neq 0$$

$$Q_3 = 72 \times 10^3\, xy^3 (112x^4 + 14x^2 y^2 + y^4) \neq 0$$

are all different from zero, the theorem is applicable as it stands. Also, since $E_0^{(k)} = 0$ for $k=1,2,3,4$, we conclude that (56) is integrable, admitting a *genuine quartic* with $e_0 = 1$ and all the other constants (16*) zero.

Having proved that the potential function (56) is integrable we now proceed to find the corresponding quartic integral. To this end we follow the steps (vi)-(viii) of section 8. With

$$A=B=\Gamma=\Delta=0 \qquad E=1$$

we find

$$h_0=0$$

and

$$I=-\tfrac{4}{3}x^2y^4-\tfrac{2}{9}y^6$$

$$F=0 \qquad G=-\tfrac{4}{3}y^3 \qquad H=4xy^2.$$

Thus the integral (9) is written as

$$\varphi=\dot{y}^4-\tfrac{4}{3}y^3\dot{x}\dot{y}+4xy^2\dot{y}^2-\tfrac{4}{3}x^2y^4-\tfrac{2}{9}y^6.$$

(b) As a second example we take

$$U=2\sin x\sin y. \tag{57}$$

We shall see how the theory developed in this paper can detect the fact that (57) is integrable, admitting a pseudo-quartic constant of motion.

We first obtain from (22) $J=0$. We understand that the theorem of section 7 is not applicable as it stands and the case must be handled according to section 10. From (50) we find that $\delta^*=0$, so we are in case (ii) of section 10 and proceed as follows: we check that $K_2L-L_2K=0$ for any values of the constants (16*). We then have to find if, for adequate constants (16*), there exists a function $h(x, y)$ satisfying equations (52a) and (49a), which, for (57), are written as follows:

$$h_x=-4AU_{10}-BU_{01} \tag{58a}$$

$$\begin{aligned}3\cos x\sin y\cos y\,h_{yy}+\cos x\,h_y &= -6\cos^2 x\sin y(1+\cos^2 y)B-4\cos^2 x\sin y\cos^2 y\,D\\ &\quad -40\sin x\cos x\cos^3 y\,E+6\cos^2 x\sin^2 y\cos y\,C_x\\ &\quad -6\sin x\cos x\sin y\cos^2 y\,D_x-30\sin^2 x\cos^3 y\,E_x.\end{aligned} \tag{58b}$$

Since C is absent from both equations (58) we set

$$A=B=\Delta=E=0 \qquad C=1 \tag{59}$$

(i.e. $c_0=1$ and the remaining constants (16*) zero) and observe that (58) has the obvious solution $h=0$.

Integrability will be made sure if, for $h=0$, we manage to find a solution of the system (19), which, for the case at hand, is

$$I_{xx}=\frac{\cos x}{\sin x}I_x+8\sin^2 x\cos^2 y \tag{60a}$$

$$I_{xy}=-\frac{(3\sin^2 x+\cos^2 x)\sin y}{3\sin^2 x\cos y}I_x+\frac{\cos x}{3\sin x}I_y+8\sin x\cos x\sin y\cos y \tag{60b}$$

$$\begin{aligned}I_{yy}=&\frac{(3\sin^2 x-\cos^2 x\sin^2 y)\cos x}{3\sin^3 x\cos^2 y}I_x\\ &+\frac{(\cos^2 x-3\sin^2 x)\sin y}{3\sin^2 x\cos y}I_y+8\cos^2 x\sin^2 y.\end{aligned} \tag{60c}$$

Solving (60a) for I_x we obtain

$$I_x = C(y)\sin x - 8\sin x\cos x\cos^2 y \tag{61a}$$

where $C(y)$ is an arbitrary function of y. Inserting (61a), into (60b) we find

$$I_y = \frac{3\sin^2 x}{\cos x}C(y) + \frac{(3\sin^2 x + \cos^2 x)\sin y}{\cos x\cos y}C(y) - 8\cos^2 x\sin y\cos y. \tag{61b}$$

The compatibility condition for (61) determines

$$C(y) = C_0\cos y \tag{62}$$

where C_0 is a constant. It is easily shown that

$$I = -C_0\cos x\cos y + 4\cos^2 x\cos^2 y \tag{63}$$

found from (61), satisfies also equation (60c). From (17) we find, in view of (59) and (63),

$$g(x, y, z) = \left(\frac{C_0}{2} - 4\cos x\cos y\right)z$$

and from (11b)

$$F = 0 \qquad G = -\frac{C_0}{2} + 4\cos x\cos y \qquad H = 0. \tag{64}$$

In view of (59), (64) and (63) we write the second integral (9)

$$\varphi = \dot{x}^2\dot{y}^2 + 4\cos x\cos y\,\dot{x}\dot{y} + 4\cos^2 x\cos^2 y - \tfrac{1}{2}C_0(\dot{x}\dot{y} + 2\cos x\cos y). \tag{65}$$

The constant C_0 in (65) is actually superfluous. It is easily seen that (65) is a pseudo-quartic expressing the constancy of

$$\varphi_2 = \dot{x}\dot{y} + 2\cos x\cos y$$

because

$$\varphi = \varphi_2^2 - \tfrac{1}{2}C_0\varphi_2.$$

(c) As a counter-example we take

$$U = x\,e^y. \tag{66}$$

We find

$$J = 2\,e^{3x} \qquad \delta = 48\,e^{14x} \qquad M_3 = -6y\,e^{7x} \qquad Q_3 = -18\,e^{7x}$$

all different from zero. Applying the theorem for $k = 1$ and ordering the result in powers of x, y we find very easily that (38) is satisfied only if all the 15 constants (16*) are zero. There is no need to check (38) for $k = 2$, 3 or 4. The potential function (66) does not accept an algebraic integral of the fourth (or second or first) degree in the velocity components.

12. Concluding remarks

We found, in Cartesian coordinates, four necessary and sufficient conditions so that a given potential function $U(x, y)$ admits an integral of motion quartic or pseudo-quartic in the velocity components $\dot{x}, \dot{y}$. The theorem requires the evaluation of the $4 \times 15 = 60$ functions

$$\begin{array}{ccccc} A_4^{(k)} & A_3^{(k)} & A_2^{(k)} & A_1^{(k)} & A_0^{(k)} \\ & B_3^{(k)} & B_2^{(k)} & B_1^{(k)} & B_0^{(k)} \\ & & C_2^{(k)} & C_1^{(k)} & C_0^{(k)} \\ & & & D_1^{(k)} & D_0^{(k)} \\ & & & & E_0^{(k)} \end{array} \tag{67}$$

for $k = 1, 2, 3, 4$. These functions are all calculated in terms of partial derivatives U_{ij} of $U(x, y)$ up to the fifth order. The calculation is straightforward, although very lengthy. It is not advisible to go into this task without the aid of a computer.

In the general case one assumes that the inequalities (40) are satisfied. If and only if there exist 15 constants (16*), not all zero (and not the triplet $a_0 = 1, c_0 = 2, e_0 = 1$ with the other 12 constant zero) such that each expression of the form (38) vanishes identically, the given potential accepts a quartic or pseudo-quartic integral.

The theorem is directly informative in cases like the first example in section 11, i.e. if, for the given $U(x, y)$, one of the coefficients (67), say E_0, is identically zero for all values of $k = 1, 2, 3, 4$. If this is not the case, it is suggested that the theorem be applied only for one k, say $k = 1$, and try to find non-zero 15-ples $(a_4, a_3, \ldots, e_0)$, if they exist. Ordering in powers of x, y makes this task very easy. In general the answer will be negative and this suffices to establish non-integrability of the sort studied in this paper. If, however, we do find non-zero constants (16*) satisfying the theorem for $k = 1$, then we can use these constants and check the remaining three conditions.

In the case of the counter-example we noticed that $E_0^{(2)} = 0$, but $E_0^{(1)} \neq 0$ and the case was characterized as non-integrable. The theorem can also detect integrability associated with quadratic integrals and, in this sense, it is a generalization of the well known result due to Darboux [1, 2]. Of course, in the case of quadratics, the condition is just one (not four) and the corresponding partial differential equation is of the second order and, what is most important, its general solution is available. The present theorem is good for checking possible integrability and, of course, the general solution for $U(x, y)$ is not known.

Although not attempted in this paper the theorem may be used for searching for potentials of a certain type (e.g. polynomials of a certain degree) associated with quartics. This is easier for quartics of a more specific form as for instance, those including only one (out of the five) term of the fourth degree, e.g. the form studied recently by Evans [13] and earlier by Bozis [1] with $\dot{x}^2\dot{y}^2$ factored by a constant.

Appendix

In terms of J and all the basic functions K and L we first prepare the functions, M, N, P, Q. They are:

$$M_3 = K_2 J$$

$$M_i = (K_{iy} - L_{ix} + K_{i-1})J + L_i J_x - K_i J_y \qquad \text{(for } i = 2 \text{ and } i = 1\text{)}$$

$$M_{0a}=J_yK_{0a}-JK_{0ay}-J_xL_{0a}+JL_{0ax}-4J(U_{12}L_2+U_{11}L_1+U_{10}L_0)$$

$$M_{0b}=J_yK_{0b}-JK_{0by}-J_xL_{0b}+JL_{0bx}-J(U_{03}L_2+U_{02}L_1+U_{01}L_0)$$

$$M_{0s}=J_yK_{0s}-JK_{0sy}-J_xL_{0s}+JL_{0sx} \qquad (\text{for } s=c,d,e)$$

$$M_{1b}=J_yK_{1b}-JK_{1by}-J_xL_{1b}+JL_{1bx}+J(K_{0a}+L_{0b})+4J(2U_{11}L_2+U_{10}L_1)$$

$$M_{1c}=J_yK_{1c}-JK_{1cy}-J_xL_{1c}+JL_{1cx}+J(K_{0b}+L_{0c})+J(2U_{02}L_2+U_{01}L_1)$$

$$M_{1s}=J_yK_{1s}-JK_{1sy}-J_xL_{1s}+JL_{1sx}+J(K_{0s^*}+L_{0s})$$

(for $s=d, e$ and $s^*=c, d$ correspondingly)

$$M_{2c}=J(K_{1b}+L_{1c})-4JU_{10}L_2$$

$$M_{2d}=J(K_{1c}+L_{1d})-JU_{01}L_2$$

$$M_{2e}=J(K_{1d}+L_{1e})$$

$$N_2=-JK_{2x}+J_xK_2+JK_2(U_{11}/U_{01})$$

$$N_1=-JK_{1x}+J_xK_1+JK_1(U_{11}/U_{01})+J^2U_{01}$$

$$N_{0a}=JK_{0ax}-J_xK_{0a}-JK_{0a}(U_{11}/U_{01})-4J(K_2U_{12}+K_1U_{11}+K_0U_{10})-8J^2U_{10}^2$$

$$N_{0b}=JK_{0bx}-J_xK_{0b}-JK_{0b}(U_{11}/U_{01})-J(K_2U_{03}+K_1U_{02}+K_0U_{01})-5J^2U_{01}U_{10}$$

$$N_{0c}=JK_{0cx}-J_xK_{0c}-JK_{0c}(U_{11}/U_{01})-2J^2U_{01}^2$$

$$N_{0s}=JK_{0sx}-J_xK_{0s}-JK_{0s}(U_{11}/U_{01}) \qquad (\text{for } s=d,e)$$

$$N_{1b}=JK_{1bx}-J_xK_{1b}-J((U_{11}/U_{01})K_{1b}-K_{0b})+4J(2K_2U_{11}+K_1U_{10})$$

$$N_{1c}=JK_{1cx}-J_xK_{1c}-J((U_{11}/U_{01})K_{1c}-K_{0c})+J(2K_2U_{02}+K_1U_{01})$$

$$N_{1s}=JK_{1sx}-J_xK_{1s}-J((U_{11}/U_{01})K_{1s}-K_{0s}) \qquad (\text{for } s=d,e)$$

$$N_{2c}=JK_{1c}-4JK_2U_{10}$$

$$N_{2d}=JK_{1d}-JK_2U_{01}$$

$$N_{2e}=JK_{1e}.$$

With μ_1 given by $(20a)$ we find

$$P_2=(\mu_1K_2-L_{2x})J+L_2(J_x+(U_{11}/3U_{01})J)$$

$$P_1=(\mu_1K_1-L_{1x})J+L_1(J_x+(U_{11}/3U_{01})J)-J^2U_{10}$$

$$P_{0a}=-(\mu_1K_{0a}-L_{0ax})J-L_{0a}(J_x+(U_{11}/3U_{01})J)-4J(U_{12}L_2+U_{11}L_1+U_{10}L_0)$$

$$P_{0b}=-(\mu_1K_{0b}-L_{0bx})J-L_{0b}(J_x+(U_{11}/3U_{01})J)-J(U_{03}L_2+U_{02}L_1+U_{01}L_0)-J^2U_{10}^2$$

$$P_{0c}=-(\mu_1K_{0c}-L_{0cx})J-L_{0c}(J_x+(U_{11}/3U_{01})J)-2J^2U_{01}U_{10}$$

$$P_{0d}=-(\mu_1K_{0d}-L_{0dx})J-L_{0d}(J_x+(U_{11}/3U_{01})J)-2J^2U_{01}^2$$

$$P_{0e}=-(\mu_1K_{0e}-L_{0ex})J-L_{0e}(J_x+(U_{11}/3U_{01})J)$$

$$P_{1b}=-(\mu_1K_{1b}-L_{1bx}-L_{0b})J-L_{1b}(J_x+(U_{11}/3U_{01})J)+4J(2U_{11}L_2+U_{10}L_1)$$

$$P_{1c}=-(\mu_1K_{1c}-L_{1cx}-L_{0c})J-L_{1c}(J_x+(U_{11}/3U_{01})J)+J(2U_{02}L_2+U_{01}L_1)$$

$$P_{1s} = -(\mu_1 K_{1s} - L_{1sx} - L_{0s})J - L_{1s}(J_x + (U_{11}/3U_{01})J) \qquad (\text{for } s = d, e)$$

$$P_{2c} = JL_{1c} - 4JU_{10}L_2$$

$$P_{2d} = JL_{1d} - JU_{01}L_2$$

$$P_{2e} = JL_{1e}.$$

With μ_3, μ_4 given by $(20c), (20d)$ we find

$$Q_3 = L_2 J$$

$$Q_2 = J(L_{2y} + L_1) - J_y L_2 - J(\mu_3 K_2 + \mu_4 L_2)$$

$$Q_1 = J(L_{1y} + L_0) - J_y L_1 - J(\mu_3 K_1 + \mu_4 L_1) - J^2(U_{10}^2/U_{01})$$

$$Q_{0a} = (\mu_4 L_{0a} + \mu_3 K_{0a})J + L_{0a}J_y - JL_{0ay}$$

$$Q_{0b} = (\mu_4 L_{0b} + \mu_3 K_{0b})J + L_{0b}J_y - JL_{0by} + J^2(U_{10}^3/U_{01})$$

$$Q_{0c} = (\mu_4 L_{0c} + \mu_3 K_{0c})J + L_{0c}J_y - JL_{0cy} + 2J^2 U_{10}^2$$

$$Q_{0d} = (\mu_4 L_{0d} + \mu_3 K_{0d})J + L_{0d}J_y - JL_{0dy} + 4J^2 U_{01} U_{10}$$

$$Q_{0e} = (\mu_4 L_{0e} + \mu_3 K_{0e})J + L_{0e}J_y - JL_{0ey} + 8J^2 U_{01}^2$$

$$Q_{1s} = (\mu_4 L_{1s} + \mu_3 K_{1s})J + L_{1s}J_y - J(L_{1sy} - L_{0s^*})$$

(for $s = b, c, d, e$ and $s^* = a, b, c, d$ correspondingly).

$$Q_{2s} = JL_{1s^*} \qquad (\text{for } s = c, d, e \text{ and } s^* = b, c, d \text{ correspondingly}).$$

We no longer need J and the functions K and L. To find the functions R for $k = 1$ we need δ, as given by (33), and also

$$\delta_1 = \delta P_{2x} - \delta_x P_2 \qquad \delta_2 = \delta N_{2x} - \delta_x N_2.$$

Then

$$R_{0a}^{(1)} = \delta_1 N_{0a} - \delta_2 P_{0a} + \delta(P_2 N_{0ax} - N_2 P_{0ax}) + 4\delta^2 U_{11}$$

$$R_{0b}^{(1)} = \delta_1 N_{0b} - \delta_2 P_{0b} + \delta(P_2 N_{0bx} - N_2 P_{0bx}) + \delta^2 U_{02}$$

$$R_{0s}^{(1)} = \delta_1 N_{0s} - \delta_2 P_{0s} + \delta(P_2 N_{0sx} - N_2 P_{0sx}) \qquad (\text{for } s = c, d, e)$$

$$R_{1b}^{(1)} = \delta_1 N_{1b} - \delta_2 P_{1b} + \delta[P_2(N_{1bx} + N_{0b}) - N_2(P_{1bx} + P_{0b})] - 4\delta^2 U_{10}$$

$$R_{1c}^{(1)} = \delta_1 N_{1c} - \delta_2 P_{1c} + \delta[P_2(N_{1cx} + N_{0c}) - N_2(P_{1cx} + P_{0c})] - \delta^2 U_{01}$$

$$R_{1s}^{(1)} = \delta_1 N_{1s} - \delta_2 P_{1s} + \delta[P_2(N_{1sx} + N_{0s}) - N_2(P_{1sx} + P_{0s})] \qquad (\text{for } s = d, e)$$

$$R_{2s}^{(1)} = \delta_1 N_{2s} - \delta_2 P_{2s} + \delta[P_2(N_{2sx} + N_{1s}) - N_2(P_{2sx} + P_{1s})] \qquad (\text{for } s = c, d, e)$$

$$R_{3s}^{(1)} = \delta(P_2 N_{2s} - N_2 P_{2s}) \qquad (\text{for } s = d, e).$$

To find the functions R for $k = 2$ we introduce

$$\delta_3 = \delta P_{2y} - \delta_y P_2 + \delta P_1 \qquad \delta_4 = \delta N_{2y} - \delta_y N_2 + \delta N_1.$$

Then

$$R_{0s}^{(2)} = \delta_3 N_{0s} - \delta_4 P_{0s} + \delta(P_2 N_{0sy} - N_2 P_{0sy}) \qquad (\text{for } s = a, b, c, d, e)$$

$$R_{1s}^{(2)} = \delta_3 N_{1s} - \delta_4 P_{1s} + \delta[P_2(N_{1sy} - N_{0s^*}) - N_2(P_{1sy} - P_{0s^*})]$$

(for $s = b, c, d, e$ and $s^* = a, b, c, d$ correspondingly)

$$R_{2s}^{(2)} = \delta_3 N_{2s} - \delta_4 P_{2s} + \delta[P_2(N_{2sy} - N_{1s^*}) - N_2(P_{2sy} - P_{1s^*})]$$

(for $s = c, d, e$ and $s^* = b, c, d$ correspondingly)

$$R_{3s}^{(2)} = -\delta(P_2 N_{2s^*} - N_2 P_{2s^*})$$

(for $s = d, e$ and $s^* = c, d$ correspondingly).

We introduce the notation

$$\lambda_1 = M_3(\delta N_{1y} - \delta_y N_1) - \delta(M_1 N_2 - M_2 N_1)$$

and

$$\lambda_2 = M_3(\delta P_{1y} - \delta_y P_1) - \delta(M_1 P_2 - M_2 P_1)$$

and obtain, for $k = 3$,

$$R_{0s}^{(3)} = \lambda_1 P_{0s} - \lambda_2 N_{0s} + M_3 \delta(N_1 P_{0sy} - P_1 N_{0sy}) - \delta^2 M_{0s}$$

(for $s = a, b, c, d, e$)

$$R_{1s}^{(3)} = \lambda_1 P_{1s} - \lambda_2 N_{1s} + M_3 \delta[N_1(P_{1sy} - P_{0s^*}) - P_1(N_{1sy} - N_{0s^*})] - \delta^2 M_{1s}$$

(for $s = b, c, d, e$ and $s^* = a, b, c, d$ correspondingly)

$$R_{2s}^{(3)} = \lambda_1 P_{2s} - \lambda_2 N_{2s} + M_3 \delta[N_1(P_{2sy} - P_{1s^*}) - P_1(N_{2sy} - N_{1s^*})] - \delta^2 M_{2s}$$

(for $s = c, d, e$ and $s^* = b, c, d$ correspondingly)

$$R_{3s}^{(3)} = -M_3 \delta(N_1 P_{2s^*} - P_1 N_{2s^*})$$

(for $s = d, e$ and $s^* = c, d$ correspondingly).

Finally, for $k = 4$, we adopt the notation

$$\lambda_3 = Q_3(M_1 N_2 - M_2 N_1) - M_3(Q_1 N_2 - Q_2 N_1)$$

$$\lambda_4 = Q_3(P_1 M_2 - P_2 M_1) - M_3(P_1 Q_2 - P_2 Q_1)$$

and write

$$R_{0s}^{(4)} = \delta(Q_3 M_{0s} - M_3 Q_{0s}) + \lambda_3 P_{0s} + \lambda_4 N_{0s} \qquad \text{(for } s = a, b, c, d, e\text{)}$$

$$R_{1s}^{(4)} = \delta(Q_3 M_{1s} - M_3 Q_{1s}) + \lambda_3 P_{1s} + \lambda_4 N_{1s} \qquad \text{(for } s = b, c, d, e\text{)}$$

$$R_{2s}^{(4)} = \delta(Q_3 M_{2s} - M_3 Q_{2s}) + \lambda_3 P_{2s} + \lambda_4 N_{2s} \qquad \text{(for } s = c, d, e\text{)}$$

$$R_{3s}^{(4)} = 0 \qquad \text{(for } s = d, e\text{)}.$$

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